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THE EFFECT OF A CHANGE IN ENTROPY ON THE FORM OF THE SHOCK ADIABATIC CURVE OF QUASI-TRANSVERSE ELASTIC WAVES†

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The shock adiabatic curve of quasi-transverse shock waves in a slightly anisotropic elastic medium is considered. For waves whose intensity is not too high, the elastic potential of the medium Φ is represented by an expansion in series in the deformations and the change in entropy. Shock waves, when only the principal terms remain in the expansion of Φ, which reveal the effects of non-linearity and anisotropy, have been investigated in detail in previous publications, but the effect of a change in entropy on the elastic properties of the medium was not taken into account. Below, these results are taken as the zeroth approximation and, using the method of linearization about the zeroth approximation, corrections due to succeeding terms of the expansion of Φ are obtained. A model of a slightly non-linear elastic medium is considered in detail when, in addition terms in the form of cross products of the deformations and the entropy and the square of the change in entropy are taken into account in the expansion of Φ. For such a medium, the changes in the form of the shock adiabatic curve and the position of the evolution parts in it, due to considering the effect of a jump in entropy in the shock wave, are obtained in explicit form.

1. INITIAL RELATIONS

The zeroth approximation. The relations on shock waves in elastic media have the form [1]

$$[\partial\Phi/\partial u_i] = \rho_0 W^2[u_i], \quad i = 1, 2, 3 \tag{1.1}$$

$$[\Phi] = -\tfrac{1}{2}[\partial\Phi/\partial u_i][u_i] + (\partial\Phi/\partial u_i)^+[u_i] \tag{1.2}$$

$$[v_i] + W[u_i] = 0 \tag{1.3}$$

Here $u_i = \partial w_i/\partial x$, $v_i = \partial w_i/\partial t$, $\partial w_i/\partial t$, w_i is the displacement vector of points of the medium, $\partial\Phi/\partial u_i$ are the components of the Piola–Kirchhoff stress tensor, and x is the Lagrange coordinate, for which we take the Cartesian coordinate normal to the front in the initial state, while the x_1 and x_2 axes lie in the plane of the wave front. Here and henceforth summation over repeated subscripts is assumed. The quantities u_i characterize the deformation of the medium for one-dimensional motions and, together with the components of the velocity v_i, experience a discontinuity on the shock wave front, ρ_0 is the density of the medium in front of the shock wave, $\Phi = \rho_0 U(u_i, S)$ is the elastic potential of the medium, i.e. the internal energy (in the actual state), referred to unity of the initial volume (before the passage of the shock wave), $U(u_i, S)$ is the internal energy per unit mass, S is the entropy of unit mass of the medium, and W is the Lagrangian propagation velocity of the shock wave $W = dx/dt$. The square brackets, as usual, denote jumps in the quantity enclosed with them: $[A] = A^+ - A^-$, where A^- is the value in front of the discontinuity and $A^+ = A$ is the value after the discontinuity.

The following approximate expression was used in [2] as the elastic potential $\Phi(u_i, S)$ when describing quasi-transverse waves of small amplitude in slightly anisotropic media

$$\Phi^0(u_i, S) = \frac{f}{2}r^2 + \frac{d}{2}u_3^2 + br^2u_3 + \frac{h}{4}r^4 + \frac{g}{2}(u_2^2 - u_1^2) + \rho_0 T_0(S - S_0), \quad r^2 = u_1^2 + u_2^2 \tag{1.4}$$

Here f, g, d, b and h are constant coefficients of the expansion, which have the meaning of the elasticity constants of the medium, and it is assumed that the coefficient g, corresponding to the anisotropy, is

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much less than the other coefficients (if we assume that the values of the deformations and their changes are of the order of ε, the effects of the anisotropy and non-linearity on the stress will be of the same order, if $g \sim \varepsilon^2$). The quantities $T_0 = T^-$, $S^0 = S^-$ and $\rho_0 = \rho^-$ correspond to the state before the discontinuity. In expression (1.4) only terms of higher order than the second remain; these are important for describing the principal non-linear effects in quasi-transverse shock waves. There are no terms here of higher order infinitesimals, in particular, terms containing the product of the change in entropy by some function of u_i, responsible for the effect of a change in entropy on the mechanical properties of the medium. There are also no anisotropic terms containing u_i at higher powers than the second (a consequence of the assumed smallness of the anisotropy) and no terms containing $u_1^2 + u_2^2$ in powers higher than the second (due to the assumed weak non-linearity).

We will call the relations on the discontinuity (1.1) with elastic potential $\Phi = \Phi^0$ and consequences of them the zeroth approximation. In particular, in this approximation we have the following equality for quasi-transverse waves for all u_i and u_2 [2, 3]

$$[u_3] = -\frac{b}{d-f}[r^2] \tag{1.5}$$

It enables us to eliminate u_3 from the first two relations of (1.1) and to consider them separately, assuming that Φ^0 is a function of u_1 and u_2. Here, Eqs (1.1)–(1.3) retain their form, except that we must put $i = 1, 2$ and the function Φ^0 must replaced by

$$H(u_1, u_2, S) = \frac{f}{2}r^2 + \frac{g}{2}(u_2^2 - u_1^2) - \frac{\kappa}{4}r^4 + \rho_0 T_0(S - S_0), \quad \kappa = \frac{2b^2}{d-f} - h \tag{1.6}$$

The shock adiabatic curve has been obtained and investigated in this zeroth approximation in [2, 3]. Its projection onto the (u_1, u_2) plane has the form

$$\begin{gathered} \mathcal{F}_0(u_1, u_2) \equiv (r^2 - R^2)(U_1 u_2 - U_2 u_1) + 2\frac{g}{\kappa}(u_1 - U_1)(u_2 - U_2) = 0 \\ U_\alpha = u_\alpha^-, \quad R^2 = U_1^2 + U_2^2, \quad \alpha = 1, 2 \end{gathered} \tag{1.7}$$

Expressions have been obtained in the same approximation for the change in entropy in quasi-transverse shock waves and for the velocity of the shock wave

$$[S]^0 = -\frac{\kappa}{4\rho_0 T_0}(r^2 - R^2)\tilde{u}^2, \quad \rho_0 W_0^2 - f + D_0(u_1, u_2) = 0 \tag{1.8}$$

where

$$D_0(u_1, u_2) = -\kappa\{r^2 - R^2 + U_1 u_1 + U_2 u_2 +$$

$$+ \frac{1}{\tilde{u}^2}\left(\frac{g}{\kappa}((u_1 - U_1)^2 - (u_2 - U_2)^2) + 2(U_1(u_1 - U_1) + U_2(u_2 - U_2))^2\right)\}$$

$$\tilde{u}^2 = ((u_1 - U_1)^2 + (u_2 - U_2))^2$$

We must substitute into these expressions values of u_1 and u_2 which satisfy equality (1.7). The quantity $[S]^0$ turned out to be a fourth-order infinitesimal (in ε), and hence in the zeroth approximation its effect on the properties of the shock adiabatic curve was ignored. This enabled us to consider the first group of relations on the discontinuity (1.1) independently of the energy equation (1.2), a consequence of which is the first of equalities (1.8).

The requirements that the entropy should be non-decreasing must be satisfied on the shock waves

$$S - S_0 \geq 0 \tag{1.9}$$

and also evolution conditions, which impose limitations on the relation between the velocity W of the jump and the velocities $c_\alpha^\pm$ of small perturbations. For waves travelling in the positive direction of the x axis, the evolution conditions for quasi-transverse shock waves take the form of two systems of inequalities, corresponding to two types of waves

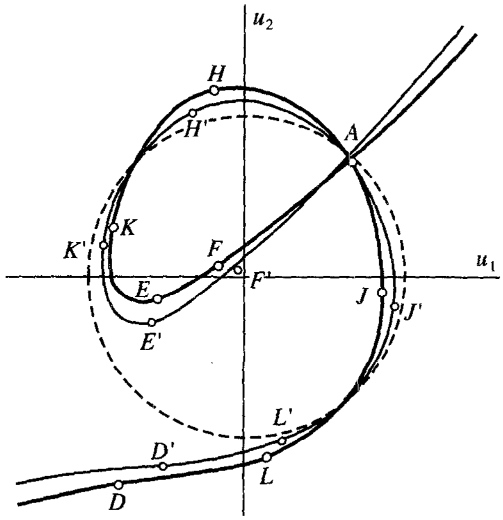

Fig. 1

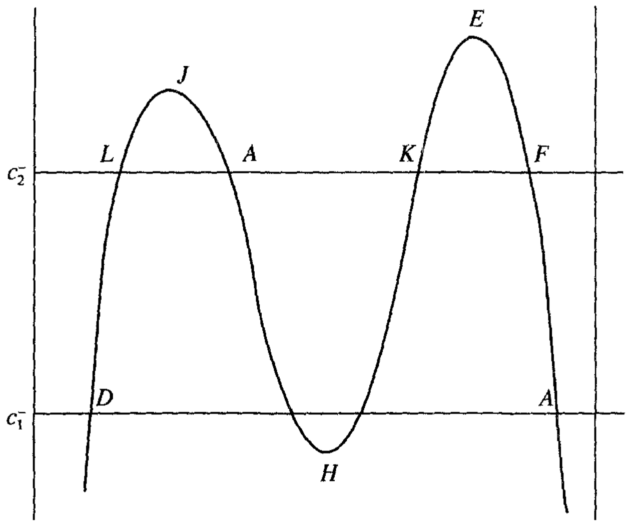

Fig. 2

$$\begin{aligned} &\text{a) } c_1^- \le W \le c_2^-, \quad 0 < W \le c_1^+ \quad \text{for slow waves} \\ &\text{b) } c_2^- \le W, \quad c_1^+ \le W \le c_2^+ \quad \text{for fast waves} \end{aligned} \tag{1.10}$$

Here c_α^- and c_α^+ ($\alpha = 1, 2$) are the velocities of small perturbations in front of and behind the discontinuity.

The simultaneous satisfaction of conditions (1.9) and (1.10) distinguish sections on the shock adiabatic curve (1.7), which can be used to construct solutions (the evolution sections). In Fig. 1 the thick curve shows the shock adiabatic curve of the zeroth approximation in the plane of shear deformations (u_1, u_2). The state in front of the discontinuity is represented by the point A with coordinates (U_1, U_2), and the dashed curve is a circle on which according to the first equality of (1.8) $S = S_0$, where $S \geqslant S_0$ inside the circle for media with $\kappa = 0$, and outside the circle for media with $\kappa < 0$. The letters F, K, E, J, D, L and H on the shock adiabatic curve indicate the position of the Jouguet points (at which $W = c_\alpha^\pm$), which are the boundaries of the evolution sections. Note that both the form of the shock adiabatic curve and the position of the Jouguet points on it do not depend on the sign of κ. However, to fix our ideas, we will henceforth assume that $\kappa > 0$.

In Fig. 2 we show a sample graph of the velocity of the jump W_0 in a medium with $\kappa > 0$ as a function of a certain parameter l, which varies monotonically along the shock adiabatic curve, for the zeroth approximation. The Jouguet points F, K, D and L, for the state in front of the discontinuity, lie at the intersection of the graph $W_0(1)$ with the horizontal lines $\bar{c}_1$ and $\bar{c}_2$. The extremum points of the velocity E, J and H are the Jouguet points for the state behind the discontinuity. The function $S(l)$ also has an extremum at the these points [2, 4, 5]. The corresponding points are denoted by the same letters in Figs 1 and 2. The two points in Fig. 2, representing the initial state, correspond to the point A in Fig. 1.

2. THE FIRST APPROXIMATION, GENERAL FORMULAE

We will now obtain a refinement of the form of the shock adiabatic curve and the positions of the Jouguet points on it for a more complete representation of the elastic potential, while continuing to assume that the deformations are not too large. The problem consists of taking into account the next terms of the expansion of the elastic potential of the medium $\Phi(u_1, u_2, u_3, S)$ compared with those written in relation (1.4). We will put

$$\Phi = \Phi^0 + \Phi^1, \quad \Phi^1 \ll \Phi^0 \tag{2.1}$$

The function Φ^0 is defined by equality (1.4), while Φ^1 represents small corrections, the effect of which it is required to take into account. We will assume that the strong inequality between Φ^0 and Φ^1, postulated in (2.1), is also satisfied for the first and second derivatives of these functions.

To investigate the projection of the shock adiabatic curve onto the u_i, u_2 plane, we used the third ($i = 3$) equation of (1.1) to eliminate $[u_3]$ from the first two. We will write this equation in the form

$$[u_3] + \frac{b}{d-f}[r^2] = q_3, \quad q_3 = \frac{b(\rho_0 W^2 - f)}{(d-f)^2}[r^2] - \frac{1}{d-f}\left[\frac{\partial \Phi^1}{\partial u_2}\right] \tag{2.2}$$

The first two ($i = 1, 2$) equations of (1.1) are given in the form

$$\left[\frac{\partial H}{\partial u_\alpha}\right] - \rho_0 W^2 [u_\alpha] = q_\alpha, \quad q_\alpha = -b u_\alpha q_3 - \left[\frac{\partial \Phi^{(1)}}{\partial u_\alpha}\right], \quad \alpha = 1, 2 \tag{2.3}$$

The function H is represented by equality (1.6). The left-hand sides of Eqs (2.2) and (2.3) correspond to the zeroth approximation. The right-hand sides arise from adding Φ^1 to the function Φ^0, and are assumed to be small. Instead of $[u_3]$ and $[S]$ on the right-hand sides we must substitute expressions (1.5) and (1.8). In this case q_α are functions of the initial state U_1, U_2, U_3 and S_0, and also of the coordinates u_1 and u_2, which specify the state behind the discontinuity. In view of the smallness of q_α, the quantities u_1 and u_2 in these expressions can be taken from the unperturbed shock adiabatic curve. When investigating the shock adiabatic curve, with the state in front of the discontinuity being given, q_α are functions of the variables u_1 and u_2, representing the state behind discontinuity.

Using the smallness of q_α, we can linearize the left-hand sides of Eqs (2.3). We obtain

$$\begin{aligned} &(H_{\alpha\beta} - \rho_0 W^2 \delta_{\alpha\beta})\delta u_\beta = \rho_0 [u_\alpha]\delta W^2 + q_\alpha, \quad \alpha, \beta = 1, 2 \\ &H_{\alpha\beta} = \partial^2 H / \partial u_\alpha \partial u_\beta \end{aligned} \tag{2.4}$$

Here δu_α are the changes in u_α which occur due to the fact that the right-hand sides of Eqs (2.3) are non-zero, and δW are the changes in the velocity of the shock wave, which are assumed to be a small arbitrary quantity. All the quantities, not indicated by the symbol δ, are assumed to be known from the zeroth approximation.

The position and form of the shock adiabatic curve in the u_1, u_2 plane for the refined model can be represented using the displacement vector $\delta\mathbf{u}$ of points of the shock adiabatic curve of the zeroth approximation. The displacement of the points $\delta\mathbf{u}$ can be considered as the sum of two vectors $\delta\mathbf{u}^{(1)} + \delta\mathbf{u}^{(2)}$ which originate from the two terms on the right-hand side of Eq. (2.4).

If we put $q_\alpha = 0$ in these equations, the difference δu_α from zero is due solely to the change in W^2 along the shock adiabatic curve of the zeroth approximation. In this case the vector $\delta\mathbf{u} = \delta\mathbf{u}^{(1)}$ turns out to be tangential to this shock adiabatic curve, and its components $\delta u_\alpha^{(1)}$ are found from the equations

$$(H_{\alpha\beta} - \rho_0 W^2 \delta_{\alpha\beta})\delta u_\beta^{(1)} = \rho_0 [u_\alpha]\delta W^2 \tag{2.5}$$

If $\delta W^2 = 0$, we obtain $\delta\mathbf{u} = \delta\mathbf{u}^{(2)}$. The displacement $\delta\mathbf{u}_i^{(2)}$ of the points of the shock adiabatic curve, corresponding to the same values of W (i.e. when $\delta W^2 = 0$), can be found from the equations

$$(H_{\alpha\beta} - W^2 \delta_{\alpha\beta})\delta u_\beta^{(2)} = q_\alpha, \quad \alpha = 1, 2 \tag{2.6}$$

In Eqs (2.4)–(2.6) it is convenient to change to a system of coordinates v_i with origin at the point of the shock adiabatic curve considered, and with axes parallel to the eigenvectors of the matrix $H_{\alpha\beta}$ at this point (which touch the integral curves of the Riemann waves). Here the components of the vectors $[u_\alpha]$, δu_α, q_α are subject to an orthogonal transformation and will be denoted by $[v_\alpha]$, v_α, h_α respectively. The matrix $H_{\alpha\beta}$ after the transformation becomes a diagonal matrix with squares of the characteristic velocities c_1^2 and c_2^2 along the principal diagonal. Equations (2.4) take the form

$$(c_{(\alpha)}^2 - W^2)\delta v_\alpha = [v_\alpha]\delta W^2 + h_\alpha, \quad \alpha = 1, 2 \tag{2.7}$$

Here, on the left-hand side, the number α on the characteristic velocity is taken in brackets in order to emphasize that summation is not carried out with respect to this subscript.

We obtain the vector $\delta\mathbf{u}^{(1)}$, tangential to the shock adiabatic curve, when $h_1 = h_2 = 0$. Its components $v_\alpha^{(1)}$ in the new system of coordinates are

$$\tau_\alpha = v_\alpha^{(1)} = \frac{[v_\alpha]}{c_\alpha^2 - W^2}\delta W^2, \quad \alpha = 1, 2 \tag{2.8}$$

When $\delta W^2 = 0$, we obtain the components $v_\alpha^{(2)}$ o the vector $\delta\mathbf{u}^{(2)}$

$$v_\alpha^{(2)} = \frac{h_\alpha}{c_\alpha^2 - W^2}, \quad \alpha = 1, 2 \tag{2.9}$$

At the Jouguet points, where $W = c_\alpha$ (c_α is the characteristic velocity behind the discontinuity), W reaches an extremum. If δW^2 is assumed to be proportional to the change in W^2 along a certain length dl of the shock adiabatic curve, then $c_\alpha^2 - W^2$ and δW^2 will simultaneously change sign on passing through a Jouguet point. We can normalize the tangential vector τ so that $|\tau| = 1$, by choosing an appropriate value of $|\delta W^2|$ (which will not be small when far from the Jouguet points).

In order to investigate on what side of the shock adiabatic curve of the zeroth approximation the refined shock adiabatic curve lies, we will write the vector product of the vectors (2.9) and (2.8) (which is directed along the u_3 axis)

$$(\mathbf{v}^{(2)} \times \boldsymbol{\tau})_3 = (\mathbf{h} \times [\mathbf{v}])_3 \frac{\delta W^2}{(c_1^2 - W^2)(c_2^2 - W^2)} \tag{2.10}$$

If we assume that the sign of δW^2 is the same as the sign of the increment of W^2 when one moves along the shock adiabatic curve in a certain direction, then, according to the above, the sign of the coefficient of the vector product of the vectors **h** and **v** will not change on crossing Jouguet points. Hence, the sign of the vector product (2.10) is determined by the sign of the vector product $\mathbf{h} \times [\mathbf{v}]$. The latter can also be written in the original system of coordinates

$$\rho_0(\mathbf{h} \times [\mathbf{v}])_3 = (\mathbf{q} \times [\mathbf{u}])_3 = q_1[u_2] - q_2[u_1] \tag{2.11}$$

Expression (2.11) vanishes at points of intersection of the "old" and refined shock adiabatic curves.

We will obtain the change in the position of the Jouguet point. This important since the Jouguet points are the ends of the evolution sections of the shock adiabatic curve. If the Jouguet condition is satisfied in front of the discontinuity, i.e. $W^2 = (c_\alpha^-)^2$, we can separately calculate the change $\delta(c_\alpha^-)^2$, while δu_β for the Jouguet points will be found from Eq. (2.4), where we put $\delta W^2 = \delta(c_\alpha^-)^2$. In the variable v_α the result can be represented in the form

$$v_\alpha = \frac{h_\alpha + [v_\alpha]\delta(c_\alpha^-)^2}{c_\alpha^2 - W^2} \tag{2.12}$$

We will now consider the changes in the shock adiabatic curve near the Jouguet points, at which W reaches its extremum values $W = c_\alpha^+ = c_\alpha$, and also the change in the position of the Jouguet point itself. Eliminating W^2 from Eqs (2.3), we obtain a new equation of the shock adiabatic curve

$$\mathcal{F}_0(u_1, u_2) = P, \quad P = (q_1[u_2] - q_2[u_1])/\kappa \tag{2.13}$$

and a new relation for W^2

$$\rho_0 W^2 - f + D_0(u_1, u_2) = Q, \quad Q = \rho_0 \frac{q_1[u_1] - q_2[u_2]}{[u_1]^2 + [u_2]^2} \tag{2.14}$$

Equations (2.13) and (2.14) only differ in their right-hand sides from the corresponding equations of the zeroth approximation.

At the Jouguet point of the zeroth approximation with respect to the state behind the discontinuity, where $W_0 = c_0$, the shock adiabatic curve touches the integral Riemann wave curve, and hence the coordinate axes v_1 and v_2 introduced above are directed along the tangent and the normal to the shock adiabatic curve at this point, respectively. Since at the Jouguet point considered W_0 has an extremum, the derivative of the function $D_0(u_1, u_2)$ with respect to v_1 is equal to zero. This enables us, in the neighbourhood of a Jouguet point of the zeroth approximation, to represent the equation of the shock adiabatic curve (1.7) and the expression for the velocity of the discontinuity respectively in the form

$$Av_2 + \frac{1}{2}Bv_1^2 = 0, \quad \rho_0 W^2 - \rho_0 W_*^2 = av_2 + \frac{1}{2}bv_1^2$$

Here, A, B and a, b are the values of the corresponding derivatives of the functions $\mathcal{F}_0$ and D_0, and W_* is the value of the velocity at the Jouguet point in the zeroth approximation. In this case Eqs (2.13) and (2.14), which take small additional terms into account, will have the form

$$\begin{gathered} Av_2 + \frac{1}{2}Bv_1^2 + Cv_1 + P = 0, \quad \rho_0 W^2 - \rho_0 W_*^2 = av_2 + \frac{1}{2}bv_1^2 + cv_1 + Q \\ C = \partial P/\partial v_1, \quad c = \partial Q/\partial v_1 \end{gathered} \tag{2.15}$$

The quantities C, c, P and Q are small quantities, taken at the Jouguet point of the zeroth approximation. Coefficients, which are non-zero in the zeroth approximation, but which are not small, remain without change. Expressing v_2 from the first equation of (2.15) and substituting into the second, we obtain

$$\begin{gathered} v_2 = -\frac{1}{A}\left(P + \frac{1}{2}Bv_1^2 + Cv_1\right) \\ \rho_0 W^2 - \rho_0 W_*^2 = Q - \frac{aP}{A} + \left(c - \frac{a}{A}C\right)v_1 + \frac{1}{2}\left(b - \frac{a}{A}B\right)v_1^2 \end{gathered} \tag{2.16}$$

The first equation of (2.16) is a more accurate equation of the shock adiabatic curve, while the second gives the values of W in it. The position of the Jouguet point corresponds to the extremum of W^2. Its coordinates in the approximation used are

$$v_2 = -\frac{P}{A}, \quad v_1 = -\frac{cA - Ca}{bA - Ba} \tag{2.17}$$

3. THE EFFECT OF A CHANGE IN ENTROPY

In the non-linear theory of elasticity, a model is often used [6] in which, in addition to the zeroth approximation employed above in the expression for the elastic potential Φ, terms with products of the deformations and the change in entropy and also a term with a quadratic change in the entropy are taken into account, i.e.

$$\begin{aligned} \Phi &= \Phi^0 + \Phi_1, \quad \Phi_1 = \rho_0 T_0(\gamma_1 I_1 + \gamma_2 I_2)(S - S_0) + \beta T_0 (S - S_0)^2 \\ I_1 &= \varepsilon_{ii}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}, \quad \gamma_1, \gamma_2, \beta - \text{const} \end{aligned} \tag{3.1}$$

Such a refinement, for example, is required for rubber-type materials, in which the elastic deformations due to a change in entropy (temperature), may turn out to be comparable with the deformations from external forces [7]. At the same time, the incompressibility condition is satisfied for similar materials, which is also used below. For one-dimensional motions, the incompressibility condition reduces to the equality u_3 = const = 0, so that there are no longitudinal waves. In this case each of the invariants I_1 and I_2 of the deformation tensor, with the relative accuracy assumed, is proportional to the expression $r^2 = u_1^2 + u_2^2$. Hence, the elastic potential of the zeroth approximation can be represented by expression (1.6), while the refining correction Φ_1 has the form

$$\Phi_1 = \rho_0 T_0 \gamma r^2 (S - S_0) + \beta (S - S_0)^2 \tag{3.2}$$

To clarify the meaning of the coefficients we note that the thermodynamic identity $T = \partial U/\partial S$ (where $U = \Phi/\rho_0$), using expressions (3.1) and (3.2), gives

$$T = T_0[1 + \gamma r^2] + 2\beta(S - S_0)/\rho_0 \tag{3.3}$$

It can be seen from this equality that in an adiabatic process (S = const) the quantity γ determines the dependence of the temperature on the formation, and also the temperature dependence of the stresses, since $\partial\Phi/\partial u_i$ are the stresses on an area normal to the x axis. We will further assume that $\gamma > 0$, which is observed in rubber at fairly low temperatures. Since, for constant deformation, the heat flux $dQ = c_u dT$, where c_u is the heat capacity where u_i = const, and $dQ = TdS$, we obtain $dT = (T/c_u)dS$. On the other hand, differentiating Eq. (3.3) with u_i = const, we obtain $dT = (2\beta/\rho_0)dS$, whence it can be seen that the coefficient β is determined by the heat capacity for constant deformation

$$\beta = \rho_0 T_0/(2c_u)$$

Obviously $\beta > 0$.

For the model employed, linearization about the zeroth approximation enables the shock adiabatic curve to be obtained as well as refining corrections to the velocity of the jump W and to the jump in entropy $S - S_0$ in explicit form.

In the case considered

$$q_\alpha = -2\gamma\rho_0 T_0 (S - S_0) u_\alpha, \quad \alpha = 1, 2 \tag{3.4}$$

and Eqs (2.3) take the form

$$[\partial H/\partial u_\alpha] - \rho_0 W^2 (u_\alpha - U_\alpha) = -2\gamma\rho_0 T_0 (S - S_0) u_\alpha, \quad \alpha = 1, 2 \tag{3.5}$$

Eliminating W, we obtain the equation of the shock adiabatic curve (2.13), in the form

$$\mathcal{F}_0(u_1, u_2) = P, \quad P = 2\gamma\rho_0 T_0 \kappa^{-1}(S - S_0)(u_1[u_2] - u_2[u_1]) \tag{3.6}$$

Unlike the zeroth approximation, in which $\mathcal{F}_0 = 0$, the form of the shock adiabatic curve now also depends on the change in entropy. For the quantity $S - S_0$ on the right-hand side of Eq. (3.6) we can use expression (1.8) from the zeroth approximation. We obtain

$$P = -\tfrac{1}{2}\gamma \tilde{u}^2 (r^2 - R^2)(U_1 u_2 - U_2 u_1) \tag{3.7}$$

Then the refined shock adiabatic curve is represented by the dependence on only the shear deformations u_1 and u_2

$$\mathcal{F} \equiv \mathcal{F}_0 - P = 0 \tag{3.8}$$

It is shown by the thin curve in the same phase plane u_1, u_2 in Fig. 1. Both curves $\mathcal{F}_0 = 0$ and $\mathcal{F} = 0$, constructed for the initial state U_1, U_2, have the same asymptote and intersect one another at points which are symmetrical with respect to the initial point about the u_1 and u_2 axes. The mutual position of the two curves can be explained using the general rules of Section 2, But, for this special case, when the explicit form of the equations of both curves is known, it is easier to do this by comparing the mutual position of the points of intersection of these curves with the coordinate axes u_1 and u_2. Here we note that when $\gamma/\kappa < 0$, the new shock adiabatic curve has additional branches, but they are outside the limits due to the assumption that u_α is small.

The velocity W of the discontinuity can be found from the conditions on the discontinuity (3.5)

$$\begin{aligned} \rho_0 W^2 &= \rho_0 W_0^2 + Q \\ Q &= 2\gamma\rho_0 T_0 (S - S_0)\{u_1(u_1 - U_1) + u_2(u_2 - U_2)\}/\tilde{u}^2 \end{aligned} \tag{3.9}$$

To calculate the change in entropy on the jump we have the equation

$$\rho_0 T_0 (S - S_0) + \tfrac{1}{4}\kappa(r^2 - R^2)\tilde{u}^2 = -\gamma\rho_0 T_0 (S - S_0)(U_1 u_1 + U_2 u_2) + \beta(S - S_0)^2$$

When the left-hand side is equal to zero, this corresponds to the zeroth approximation (1.8), and the right-hand side corresponds to the refining corrections. Linearization with respect to $S - S_0$ gives a more accurate expression for the change in entropy in the shock wave

$$S - S_0 = -\frac{\kappa}{4T_0\rho_0}(r^2 - R^2)\tilde{u}^2\left\{1 - \frac{\gamma\rho_0}{\kappa}(U_1 u_1 + U_2 u_2)\right\} \tag{3.10}$$

As also in the zeroth approximation, $S = S_0$ on the circle passing through the initial point (U_1, U_2), with centre at the origin of coordinates (it is represented by the dashed curve in Fig. 1). In the region of variation of u_α assumed, the expression in braces in (3.10) does not change sign, and the states behind the discontinuity satisfy the condition for the entropy to increase. These states lie inside the entropy circle $r^2 = R^2$ for a medium with $\kappa > 0$ and outside this circle for media with $\kappa < 0$. The coefficient β does not occur in the expression for $S - S_0$.

The temperature behind the jump is found from the condition $T = (\delta\Phi/\partial S)_{u\alpha = \text{const}}$. We have

$$\frac{T - T_0}{T_0} = \gamma(r^2 - R^2) + 2\beta(S - S_0) = \kappa(r^2 - R^2)(\gamma\rho_0/\kappa - \beta\tilde{u}^2)$$

The term with γ corresponds to the adiabatic change in temperature, while the term with β corresponds to the change in temperature due to the jump in entropy.

For the evolution sections on the shock adiabatic curve, defined by conditions (1.10), we will investigate how their ends – the Jouguet points F, K, L and D, are shifted in the state in front of the jump ($W = c_\alpha^-$) (Fig. 1) with respect to their position in the zeroth approximation. Since $\delta c_\alpha^- = 0$ in the case considered, when finding the displacement v_α of these Jouguet points we must assume $\delta W^2 = 0$ and use formulae (2.9). We recall that the components $\{v_\alpha\}$ are measured from the point considered on the zeroth shock adiabatic curve along the tangent to the integral curve of the Riemann waves passing through this point. Two families of integral curves are represented by ovals, symmetrical about the u_1, u_2 axes, elongated along the u_2 axis, and the lines, orthogonal to them, and converting into rays at infinity [8]. As can be seen from formulae (3.4), the vector $\dot{\mathbf{q}} = \{q_\alpha\}$ is directed along the radius vector u_α on the side of the origin of coordinates (since we have assumed $\gamma > 0$). Knowing the orientation of the vector $\mathbf{q}$ with respect to the v_α axis, we can state the signs of the components h_α in formula (2.9). The signs of the denominator in this formula are determined by the evolution conditions (1.10). It follows from these that [2]

$$W^F < c_1 < c_2, \quad c_1 < W^K < c_2, \quad c_1 < c_2 < W^{L,D} \tag{3.11}$$

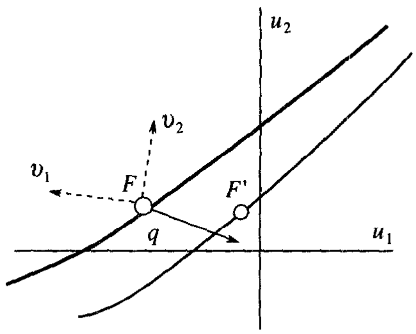

Fig. 3

We will show, as an example, how one can determine the direction of the displacement, i.e. the signs of the components v_α for Jouguet point F, representing the end of the evolution section of slow shock waves when $\kappa > 0$. In Fig. 3 we show, on an enlarged scale, the neighbourhood of the point F of the u_1, u_2 plane (Fig. 1). As before, the thick curve represents the part of the shock adiabatic curve of the zeroth approximation, the thin curve represents part of the refined shock adiabatic curve, while the dashed lines represent the elements of the integral curves of the Riemann waves, along the tangent to which at the point F the coordinate axes v_α are directed. Obviously, at the point F the vector $\mathbf{q}$ has components $h_1^F < 0, h_2^F < 0$. Taking into account the signs of the numerators in (2.9), in accordance with inequalities (3.11), we obtain $v_1^F < 0, v_2^F < 0$, which defines qualitatively the position of the point F' on the refined shock adiabatic curve. Similarly, we obtain for the other Jouguet points

$$h_1^K < 0, \quad h_2^K > 0, \quad h_1^L < 0, \quad h_2^L > 0, \quad h_1^D < 0, \quad h_2^D < 0$$

The displaced points K', L' and D' are indicated in Fig. 1. The position of these Jouguet points in the zeroth approximation and in the refined case can also be found numerically, using $W = c_\alpha$ to determine them. The expression for W is given by formula (3.9). The characteristic velocities are found by the standard method [1]

$$c_{1,2}^2 = \frac{1}{2}(\Phi_{11} + \Phi_{22}) \pm \frac{1}{2}\sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}}, \quad \Phi_{\alpha\beta} = \frac{\partial^2 \Phi}{\partial u_\alpha \partial u_\beta}$$

For the zeroth approximation

$$\rho_0 (c_{1,2}^2)^0 = f - \kappa\left(2(u_1^2 + u_2^2) \pm \sqrt{(u_1^2 - u_2^2 + g)^2 + 4u_1^2 u_2^2}\right)$$

For the additional term Φ^1 of the elastic potential we have

$$\Phi_{11}^1 = \Phi_{22}^1 = 2\gamma\rho_0 T_0 (S - S_0), \quad \Phi_{12}^1 = 0$$

and, consequently, the refined expressions for the characteristic velocities have the form

$$c_\alpha^- = (c_\alpha^-)^0, \quad c_\alpha^2 = (c_\alpha^2)^2 + 2\gamma\rho_0 T_0 (S - S_0)$$

In the phase plane u_1, u_2, the intersection of the shock adiabatic curve $\mathcal{F}_0 = 0$ with the line $W^0 = c_2^-$ gives the Jourguet points F, K and D, while the intersection with the line $W^0 = c_1^-$ gives the point L. The new positions of the corresponding points F', K', D' and L' are found from the intersection of the shock adiabatic curve $\mathcal{F} = 0$ with the lines $W = c_\alpha^- (\alpha = 1, 2)$. Numerical results confirm the qualitative conclusions reached above.

It is extremely difficult to investigate the displacements of the Jouguet points E, J and H with respect to the state behind the jump ($W = c_\alpha^+$) using formulae (2.15)–(2.17). Hence, for these points we immediately use the numerical method proposed above. The points E', J' and H' obtained in this way are shown in Fig. 1.

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